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How to Describe the Space-Time Structure with Nets of C*-Algebras

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The major subject of algebraic quantum field theory is the study of nets of local C*-algebras, i.e., maps $\mathbb{O} \mapsto \mathscr{A}(\mathbb{O})$ assigning to each open, relatively compact region $\mathbb{O}$ of space-time (M, g) a C*-algebra $\mathscr{A}(\mathbb{O})$, whose self-adjoint elements describe local observables measurable in the region $\mathbb{O}$. A question discussed recently in a number of papers is how much information about the geometric structure of the underlying space-time (M, g) is encoded in the algebraic structure of the net $\mathbb{O} \mapsto \mathscr{A}(\mathbb{O})$. Following these ideas, it is demonstrated in this paper how space-time-related concepts like causality and observers can be described in a purely algebraic way, i.e., using only the local algebras $\mathscr{A}(\mathbb{O})$. These results are then used to show how the space-time (M,g) can be reconstructed from the set $\mathscr{L}_{\text{loc}} := \{\mathscr{A}(\mathbb{O}) | \mathbb{O} \subset M \text{ open}, \bar{\mathbb{O}} \text{ compact}\}$ of local algebras.

1. INTRODUCTION

All possibilities to get information about space-time structure are based on observations and manipulations of material objects. It is therefore natural to use this idea as a base for axiomatic foundations of space and time. This is done, e.g., by Ehlers *et al.* (1972) using classical particles and light rays or Lämmerzahl (1990) with classical fields. However, classical matter models are appropriate only if the large-scale structure of space-time is considered. For microscopic scales elementary particles are needed as test particles. "Observation of material objects" means in this case the measurement of local observables of a quantum field theory. These considerations lead in a natural way to the question of how a unified axiomatic scheme for space-time and quantum field theory can be formulated.

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The aim of this paper is to present ideas on how some aspects of this question can be answered. The starting point for this purpose is the algebraic formulation of quantum field theory [see Haag (1992) for a detailed exposition]. The basic idea is to describe observables, measurable in a bounded space-time region $\mathcal{O}$ by self-adjoint elements of a C*-algebra $\mathcal{A}(\mathcal{O})$. The family $(\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{B}(M)}$ of all these local algebras is isotone: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$, hence it is a net of C*-algebras indexed by open, relatively compact subsets of space-time. The latter is described classically, i.e., by a Lorentzian manifold (M, g). If the structure of space-time, in this case the Lorentzian manifold (M, g), can be recovered completely from measurements of local observables $A \in \mathcal{A}(\mathcal{O})$, it should be possible to reduce all statements about the geometry of (M, g) to statements about the set of algebras

$$\mathcal{L}_{\text{loc}} := \{\mathcal{A}(\mathcal{O}) | \ \mathcal{O} \in \mathcal{B}(M)\} \tag{1}$$

This consideration suggests the idea to use a pair $(\mathfrak{A}_1, \mathcal{L}_{\text{loc}})$ consisting of a C*-algebra $\mathfrak{A}_1$ and a set of closed, *-subalgebras of $\mathfrak{A}_1$ [which need not necessarily have the form of equation (1)] to describe space-time as well as quantum field theory. Physically an algebra $\mathfrak{A} \in \mathcal{L}_{\text{loc}}$ describes on the one hand local observables, measurable in a distinguished space-time region, and on the other hand this region itself. This interpretation is based on the fact that it does not make much sense to make differences between two regions $\mathcal{O}_1$, $\mathcal{O}_2$ of (classical) space-time (M, g) if the corresponding algebras are identical, since it is not possible to distinguish between $\mathcal{O}_1$ and $\mathcal{O}_2$ by local measurements.

This ansatz provides a reasonable basis for space-time concepts which are more general than general relativity. In particular, we can hope to get a description of space-time for microscopic length and time scales. However, it is not clear at all which axioms the set $\mathcal{L}_{\text{loc}}$ has to fulfil such that physical concepts like causality, observers, and reference frames can be described in this new context. A first step to fill this gap is therefore a detailed study of well-known (and therefore in most cases classical) space-time models to get formulations of these physical notions in terms of $\mathcal{L}_{\text{loc}}$ which can be generalized later. Therefore we will consider in this paper pairs $(\mathfrak{A}_1, \mathcal{L}_{\text{loc}})$ which are derived from a net $(\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{B}(M)}$ as described in equation (1) and we will demonstrate the ideas outlined up to now with a discussion of causality and observers in this framework.

2. LOCAL ALGEBRAS

Let us start with a short review of some notions from algebraic quantum field theory [see Haag (1992) and Baumgärtel and Wollenberg (1992) for details]. For this purpose consider the Lorentzian manifold (M, g) and the set $\mathcal{B}(M) := \{\mathcal{O} \subset M | \mathcal{O} \text{ open}, \overline{\mathcal{O}} \text{ compact}\}$. A subset $\mathcal{B} \subset \mathcal{B}(M)$ will be called a *net index set* if $\mathcal{B}$ is a base for the topology of M and if each bounded subset $\mathcal{N} \subset \mathcal{B}$ admits a supremum (considering the inclusion relation as an ordering relation on $\mathcal{B}$). A *net of C*-algebras* is an isotone family $(\mathcal{A}(\mathcal{O}))_{\mathcal{O} \in \mathcal{B}}$ of C*-algebras, i.e., each $\mathcal{A}(\mathcal{O})$ is a C*-algebra and $\mathcal{O}_1 \subset \mathcal{O}_2$ implies for all $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{B}$ that $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ holds. The self-adjoint elements of $\mathcal{A}(\mathcal{O})$ represent, as already mentioned in the introduction, bounded observables measurable in $\mathcal{O}$. All $\mathcal{A}(\mathcal{O})$ are subalgebras of the *algebra of quasilocal observables* $\mathcal{A} := \cup_{\mathcal{O} \in \mathcal{B}(M)} \mathcal{A}(\mathcal{O})$. The net is called *additive* if $\mathcal{A}(\cup_{\mathcal{O} \in \mathcal{N}} \mathcal{O}) = C^* (\cup_{\mathcal{O} \in \mathcal{N}} \mathcal{A}(\mathcal{O}))$ holds for all bounded subsets $\mathcal{N}$ of $\mathcal{B}$ with $\sup(\mathcal{N}) = \cup_{\mathcal{O} \in \mathcal{N}} \mathcal{O}$. The net is called *causal* if $\mathcal{O}_1 \perp_g \mathcal{O}_2 \Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$ holds for all $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{B}$. Here the binary relation $\perp_g$ is given by causal independence of $\mathcal{O}_1$ and $\mathcal{O}_2$ [see O'Neill (1983), Ch. 14, for definitions and terminology concerning the causal structure of Lorentzian manifolds]:

$$\mathcal{O}_1 \perp_g \mathcal{O}_2 : \Leftrightarrow \mathcal{O}_2 \subset M \backslash (J^+(\mathcal{O}_1) \cup J^-(\mathcal{O}_1)) \tag{2}$$

An explicit example for a causal, additive net can be derived from the free scalar field on a globally hyperbolic Lorentzian manifold (M, g) (Dimock, 1980). In this case the (minimally coupled) Klein–Gordon equation admits unique advanced and retarded fundamental solutions $E^{\pm} : C_0^{\infty}(M, \mathbb{R}) \to C^{\infty}(M, \mathbb{R})$. Their difference $E := E^+ - E^-$ leads to the weakly symplectic vector space $(\mathcal{S}, \sigma)$ given by $\mathcal{S} := C_0^{\infty}(M, \mathbb{R})/\ker(E)$ and $\sigma([\psi_1], [\psi_2]) := \int_M (E\psi_1)(x)\psi_2(x)\lambda_g(x)$, where λ_g denotes the volume element defined by the metric. To each $\mathcal{O} \in \mathcal{B}(M)$ we can associate now a subalgebra $\mathcal{A}(\mathcal{O})$ of the CCR-algebra of this symplectic space by

$$\mathcal{A}(\mathcal{O}) := C^*(\{W([\psi]) | \mathrm{supp}(\psi) \subset \mathcal{O}\}) \tag{3}$$

where $W([\psi])$ denotes the Weyl element associated to $[\psi] \in \mathcal{S}$. The family of all those algebras forms a causal, additive net of C*-algebras.

3. LATTICES OF ALGEBRAS

In the introduction we have said already that the basic idea of this work is to reduce all physical statements about space-time structure to statements about the set

$$\mathcal{L}_{\mathrm{loc}} := \{\mathcal{A}(\mathcal{O}) | \mathcal{O} \in \mathcal{B}\} \tag{4}$$

where $(\mathcal{A}(\mathcal{O}))_{\mathcal{O} \in \mathcal{B}}$ is an additive, causal net of C*-algebras. If the index set is in addition small enough such that

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Leftrightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \tag{5}$$

holds, then $\mathscr{L}_{\rm loc}$ has the special property that each bounded (obviously $\mathscr{L}_{\rm loc}$ is ordered by inclusion) subset $\mathscr{T} \subset \mathscr{L}_{\rm loc}$ admits $C^*(\cup_{\mathfrak{A} \in \mathscr{T}} \mathfrak{A})$ as its supremum. Hence, if we add the quasilocal algebra and the "minimal algebra" $\cap_{\mathfrak{A} \in \mathscr{L}_{\rm loc}} \mathfrak{A}$, we get a complete lattice. This observation leads to the following definition (Keyl, n.d.-a):

Definition 3.1. A set $\mathscr{L}$ of C*-algebras partially ordered by inclusion is called an *additive lattice of C*-algebras* if the following conditions are satisfied:

(i) $\mathscr{L}$ is a complete lattice.

(ii) The maximal element $\mathfrak{A}_1$ and the minimal element $\mathfrak{A}_0$ are given by $\mathfrak{A}_1 = C^*(\cup_{\mathfrak{A} \in \mathscr{L}_{\rm loc}} \mathfrak{A})$ and $\mathfrak{A}_0 = \cap_{\mathfrak{A} \in \mathscr{L}_{\rm loc}} \mathfrak{A})$, where $\mathscr{L}_{\rm loc} := \mathscr{L} \backslash \{\mathfrak{A}_1, \mathfrak{A}_0\}$ is called the *set of local algebras.*

(iii) For each subset $\mathscr{T} \subset \mathscr{L}$ bounded by an $\mathscr{L}_{\rm loc} \ni \mathfrak{B} \neq \mathfrak{A}_1$ the supremum in $\mathscr{L}$ is given by $\sup_{\mathscr{L}} \mathscr{T} = C^*(\cup_{\mathfrak{A} \in \mathscr{T}} \mathfrak{A})$.

If a net $(\mathscr{A}(\mathbb{O}))_{\mathbb{O} \in \mathscr{B}}$ is given, we can construct $\mathscr{L}$ by using equation (4) and adding $\mathfrak{A}_1$ and $\mathfrak{A}_0$ as given in Definition 3.1(ii). This procedure leads to a lattice only if the net satisfies equation (5). However, if this condition is not satisfied, we can, in physically relevant cases, always find a smaller index set $\mathscr{B}' \subset \mathscr{B}$ such that (5) holds for $\mathbb{O}_1, \mathbb{O}_2 \in \mathscr{B}'$ [see the discussion of "reduced index sets" in Baumgärtel and Wollenberg (1992) for a method to construct $\mathscr{B}$]. Hence we can use $\mathscr{B}'$ instead of $\mathscr{B}$ in (4) to construct a lattice $\mathscr{L}$. Note that the net $(\mathscr{A}(\mathbb{O}))_{\mathbb{O} \in \mathscr{B}'}$ contains the same information as the original one, since we can reconstruct the algebras $\mathscr{A}(\mathbb{O})$ with $\mathbb{O} \in \mathscr{B}$ but $\mathbb{O} \notin \mathscr{B}'$ very easily from $(\mathscr{A}(\mathbb{O}))_{\mathbb{O} \in \mathscr{B}'}$ due to additivity. Therefore the necessity to use a restricted index set $\mathscr{B}'$ in some cases is not a restriction for the just outlined construction of a lattice from a net [for a more general and detailed discussion of these topics see Keyl (n.d.-a)].

Let us change now our point of view. Instead of constructing a lattice from a net, we will start with an additive lattice $\mathscr{L}$ of C*-algebras and ask for a space-time (M, g), an index set $\mathscr{B}$, and a map $\mathscr{B} \ni \mathbb{O} \mapsto \mathscr{A}(\mathbb{O}) \in \mathscr{L}_{\rm loc}$ such that $(\mathscr{A}(\mathbb{O}))_{\mathbb{O} \in \mathscr{B}}$ is an additive, causal net of C*-algebras from which $\mathscr{L}$ arises as just described. We have mentioned already that the latter is possible only if the index set $\mathscr{B}$ satisfies equation (5), i.e., if the map $\mathscr{A}$ is invertible. This fact leads to the following definition (Keyl, n.d.-a).

Definition 3.2. Consider an additive lattice $\mathscr{L}$ of C*-algebras and a space-time (M, g). A map $\lambda\colon \mathscr{L}_{\rm loc} \to \mathscr{B}(M)$ is called a realization of $\mathscr{L}$ on (M, g) if $\mathscr{B} := \lambda(\mathscr{L}_{\rm loc}) \subset \mathscr{B}(M)$ is a net index set and if λ is an order isomorphism from $\mathscr{L}_{\rm loc}$ to $\mathscr{B}$.

This concept is strongly related to the work of Bannier (1994), where a topological space is constructed from an ordered set of algebras. However, some problems occur if we apply his construction to our situation. In particular, we do not get injective maps λ in general. See the corresponding discussion in Keyl (1996).

Given a realization λ of a lattice $\mathscr{L}$ on a space-time (M, g), we can construct the net $(\mathscr{A}_\lambda(\mathbb{O}))_{\mathbb{O}\in\mathscr{B}}$ with $\mathscr{A}_\lambda(\mathbb{O}) := \lambda^{-1}(\mathbb{O})$. Hence we can use realizations λ to translate physical statements about (M, g) and the quantum field theory described by the net $(\mathscr{A}_\lambda(\mathbb{O}))_{\mathbb{O}\in\mathscr{B}}$ into statements in terms of $\mathscr{L}$ alone. We will demonstrate this in the next two sections with two particular examples.

4. CAUSALITY

One way to describe the causal structure of a space-time (M, g) is the binary relation $p \perp_g q$, $p, q \in M$ on M, which is defined by: there is no causal curve from p to q. It is easy to see that this relation is uniquely fixed by the conformal equivalence class of g $(= \{e^f g | f \in C^\infty(M, \mathbb{R})\})$. On the other hand, the converse is true as well: the conformal equivalence class is uniquely determined by $\perp_g$, and due to the close relation between the conformal and differentiable structure of Lorentz manifolds, even the differentiable structure of M is fixed by $\perp_g$ (see Hawking *et al.*, 1975; Keyl, 1996; and references therein). It is therefore an interesting question whether $\perp_g$ can be translated into a relation in a lattice of C*-algebras describing a quantum field theory on (M, g).

The first step in this direction is the relation $\perp_g \subset \mathscr{B}(M) \times \mathscr{B}(M)$ defined in equation (2). It is related to the relation $\perp_g$ between points by $\mathbb{O}_1 \perp_g \mathbb{O}_2 \Leftrightarrow p \perp_g q \ \forall p \in \mathbb{O}_1 \ \forall q \in \mathbb{O}_2$. Hence $\perp_g \subset \mathscr{B}(M) \times \mathscr{B}(M)$ can be recovered from $\perp_g \subset M \times M$ and vice versa (this justifies our use of the same notation).

Consider now a causal, additive net $(\mathscr{A}(\mathbb{O}))_{\mathbb{O}\in\mathscr{B}(M)}$ on a space-time (M, g) and the corresponding lattice $\mathscr{L}$ [constructed according to equation (4) with an appropriate index set $\mathscr{B} \subset \mathscr{B}(M)$]. The problem is to find a binary relation $\perp_c$ on $\mathscr{L}$ such that $\mathscr{A}(\mathbb{O}_1) \perp_c \mathscr{A}(\mathbb{O}_2)$ holds iff $\mathbb{O}_1 \perp_g \mathbb{O}_2$ is satisfied. Since the net $(\mathscr{A}(\mathbb{O}))_{\mathbb{O}\in;\ \mathrm{SB}(\mathscr{M})}$ is causal, a possible candidate for $\perp_c$ is

$$\mathfrak{A}_1 \perp_a \mathfrak{A}_2 :\Leftrightarrow [\mathfrak{A}_1, \mathfrak{A}_2] = \{0\} \tag{6}$$

However, there are physically relevant cases (e.g., massless fields on Minkowski space) where $\perp_a$ is not the correct choice. Nevertheless $\perp_a$ is useful for our purposes. The idea is not to compare $\perp_g$ directly with $\perp_a$, but certain hull operators associated to them. To define them, let us consider first the complementations $M \supset \mathbb{O} \mapsto \mathbb{O}^{\perp_g} \subset M$ and $\mathfrak{A}_I \supset \mathfrak{A} \mapsto \mathfrak{A}^{\perp_g} \subset \mathfrak{A}_I$ associated to $\perp_g$ and $\perp_a$, which are uniquely determined by $\mathbb{O}_1 \perp_g \mathbb{O}_2 \Leftrightarrow \mathbb{O}_1 \subset \mathbb{O}_2^{\perp_g}$ and

$\mathfrak{A}_1 \perp_a \mathfrak{A}_2 \Leftrightarrow \mathfrak{A}_1 \subset \mathfrak{A}_2^{\perp_a}$. They give rise to the hull operators $\mathcal{B}(M) \ni \mathbb{O} \mapsto \mathbb{O}^{\perp_g \perp_g} \subset M$ and $\mathcal{L} \ni \mathfrak{A} \mapsto \mathfrak{A}^{\perp_a \perp_a} \subset \mathfrak{A}_{\mathrm{I}}$. The idea is to use causally closed sets

$$\mathcal{B}^{\mathrm{cc}}(M, g) := \{\mathbb{O} \in \mathcal{B}(M) | \mathbb{O}^{\perp_g \perp_g} = \mathbb{O}\} \tag{7}$$

and algebras

$$\mathcal{L}_{\mathrm{loc}}^{\mathrm{cc}} := \{\mathfrak{A} \in \mathcal{L}_{\mathrm{loc}} | \mathfrak{A}^{\perp_a \perp_a} = \mathfrak{A}\} \tag{8}$$

and to consider only those models which have the property that $\mathbb{O} \in \mathcal{B}^{\mathrm{cc}}(M, g)$ holds iff $\mathcal{A}(\mathbb{O}) \in \mathcal{L}_{loc}^{cc}$ is satisfied [this class contains all models derived from free scalar fields according to equation (3); see Keyl (1996)]. In this case we can try to characterize $\perp_c$ by the associated hull operator (defined in the same way as $(\cdot)^{\perp_a \perp_a}$), which should coincide with $(\cdot)^{\perp^a \perp^a}$. Let us summarize the discussion up to now in the following definition:

Definition 4.1. Consider an additive lattice $\mathcal{L}$ of C*-algebras, a space-time (M, g). A realization λ of $\mathcal{L}$ on (M, g) is called *causally admissible* if $\lambda(\mathcal{L}_{loc}^{cc}) = \mathcal{B}^{\mathrm{cc}}(M, g)$. A lattice for which such a realization exists will be called *causally simple* in the following.

The basic fact about a causally simple lattice is that there is essentially only one causally admissible realization on it. To make this statement more precise, let us define *conformal equivalence* of realizations:

Definition 4.2. Two realizations λ_1, λ_2 of an additive lattice of C*-algebras $\mathcal{L}$ on the space-times (M_1, g_1) and (M_2, g_2) are called *conformally equivalent* if there is a conformal transformation $f: M_1 \to M_2$ from (M_1, g_1) to (M_2, g_2) such that $f(\lambda_1(\mathfrak{A})) = \lambda_2(\mathfrak{A})$ for all $\mathfrak{A} \in \mathcal{L}_{loc}$.

With these notions we can formulate the following theorem (Keyl, n.d.-a):

Theorem 4.3. All causally admissible realizations of a causally simple lattice of C*-algebras are mutually conformally equivalent.

This theorem says that on a causally simple lattice of C*-algebras we can define the binary relation $\perp_c$ simply by $\mathfrak{A}_1 \perp_c \mathfrak{A}_2. :\Leftrightarrow \lambda(\mathfrak{A}_1) \perp_g \lambda(\mathfrak{A}_2)$, where λ is a causally admissible realization of $\mathcal{L}$. Obviously this definition depends only on $\mathcal{L}$ and not on λ. Therefore it must be possible to characterize this relation without using a causally admissible realization λ. For this purpose we have to introduce two additional notions. The first one is the orthogonality relation $\mathfrak{A}_1 \perp_o \mathfrak{A}_2 :\Leftrightarrow \inf_{\mathcal{L}}\{\mathfrak{A}_1, \mathfrak{A}_2\} = \mathfrak{A}_0$, which depends only on the order structure of $\mathcal{L}$ (and not on the algebraic structure, in contrast to $\perp_a$). The second one is related to $\mathcal{L}_{\mathrm{loc}}^{cc}$ by the following proposition (Keyl, n.d.-a):

Proposition 4.4. Consider an additive lattice $\mathcal{L}$ of C*-algebras. Then the set $\mathcal{L}^{\mathrm{cc}} := \mathcal{L}_{loc}^{\mathrm{cc}} \cup \{\mathfrak{A}_{\mathrm{I}}, \mathfrak{A}_0\}$ with $\mathcal{L}_{\mathrm{loc}}^{\mathrm{cc}}$ defined as in equation (8) is a complete lattice, but not a sublattice of $\mathcal{L}$. The supremum in $\mathcal{L}^{cc}$ is given by $\sup_{\mathcal{L}^{cc}} \{\mathfrak{A}_1, \mathfrak{A}_2\} = (\mathfrak{A}_1 \cup \mathfrak{A}_2)^{\perp_a\perp_a}$.

Now we can state the following theorem (Keyl, n.d.-a):

Theorem 4.5. Consider a causally simple, additive lattice $\mathcal{L}$ of C*-algebras and a causally admissible realization of $\mathcal{L}$ on the space-time (M, g). Then the relation $\perp_c \subset \mathcal{L} \times \mathcal{L}$ with $\mathfrak{A}_1 \perp_c \mathfrak{A}_2 \Leftrightarrow \lambda(\mathfrak{A}_1)\perp_g \lambda(\mathfrak{A}_2)$ is given by $(\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{L}^{cc}_{loc})$

$$\begin{aligned}\mathfrak{A}_1\perp_c\mathfrak{A}_2 :\Leftrightarrow\ & \mathfrak{A}_1\perp_o\mathfrak{A}_2 \\ & \wedge(\mathfrak{A}_3 \subset \sup_{\mathcal{L}_{cc}}\{\mathfrak{A}_4 \cup \mathfrak{A}_5\} \Rightarrow \neg(\mathfrak{A}_3\perp_o\mathfrak{A}_4 \vee \mathfrak{A}_3\perp_o\mathfrak{A}_5) \\ & \forall\mathfrak{A}_3 \in \mathcal{L}_{cc}\forall\mathfrak{A}_4 \in \mathcal{L}_{cc},\ \mathfrak{A}_4 \subset \mathfrak{A}_1\ \forall\mathfrak{A}_5 \in \mathcal{L}_{cc},\ \mathfrak{A}_5 \subset \mathfrak{A}_2)\end{aligned} \quad (9)$$

if $\sup_{\mathcal{L}} \{\mathfrak{A}_1, \mathfrak{A}_2\} \neq \mathfrak{A}_I$ holds.

The most interesting consequence of this theorem is that the definition of $\perp_c$ given in (9) makes sense even if $\mathcal{L}$ is not causally simple. Therefore we have a possibility to speak about causality without explicit reference to Lorentzian manifolds or related concepts (such as causal spaces).

5. OBSERVERS

Now we want to discuss a second physical concept related to space-time: observers. In general relativity an observer is described by its parametrized worldline, i.e., a smooth, future-pointing, timelike curve $\gamma: (a, b) \to M$, where the parametrization of γ is given by the observer's clock. If he measures some quantum observables in the time interval $(t_1, t_2) \subset (a, b)$, they must be in the local algebra

$$\mathcal{A}(I^+(\gamma(t_1)) \cap I^-(\gamma(t_2))) =: \mathcal{A}_\gamma(t_1, t_2) \quad (10)$$

This construction leads to a subnet $\mathcal{A}_\gamma(t_1, t_2)$, $(t_1, t_2) \subset (a, b)$ of the net $(\mathcal{A}(\mathbb{O}))_{\mathbb{O}\in\mathcal{B}(M)}$ and the question we want to ask in this section is whether the curve γ is uniquely determined by this subnet. To reformulate this statement using the language developed in the last two sections, let us consider a causally simple, additive lattice $\mathcal{L}$ of C*-algebras and a causally admissible realization λ. The problem is then to distinguish those subsets $\mathcal{T}$ of $\mathcal{L}$ which admit a smooth (or at least continuous) timelike curve γ such that $\lambda(\mathcal{T}) = \{\mathcal{A}_\lambda(t_1, t_2) \mid (t_1, t_2) \subset (a, b)\}$. A useful tool for this purpose is the lattice $\mathcal{L}^{cc}$ introduced in Proposition 4.4.

Definition 5.1. A subset $\mathcal{T}$ of an additive lattice of C*-algebras $\mathcal{L}$ is called an *observer* if the following conditions are satisfied:

(i) $\mathcal{T}$ is, as an ordered set, isomorphic to ($a \in \{-\infty\} \cup \mathbb{R}, b \in \mathbb{R} \cup \{\infty\}$)

$$\mathcal{T}(a, b) := \{(t_1, t_2) \subset \mathbb{R} | a \leq t_1 \leq t_2 \leq b\} \tag{11}$$

An order isomorphism κ from $\mathcal{T}$ to $\mathcal{T}(a, b)$ is called a *parametrization* of $\mathcal{T}$.

(ii) $\mathcal{T}$ is a sublattice of $\mathcal{L}^{cc}$ with $\mathfrak{A}_0$ as minimal and κ^{-1} (a, b) as maximal element.

(iii) For all $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{T}$ with $\mathfrak{A}_1 \perp_o \mathfrak{A}_2$ and all $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{cc}_{loc}$ with $\mathfrak{B}_i \subset \mathfrak{A}_i$ we have

$$\mathfrak{A} \subset \sup_{\mathcal{L}^{cc}}\{\mathfrak{A}_1, \mathfrak{A}_2\} \Rightarrow \mathfrak{A} \subset \sup_{\mathcal{L}^{cc}}\{\mathfrak{B}_1, \mathfrak{B}_2\} \tag{12}$$

for all $\mathfrak{A} \in \mathcal{T}$ with $\mathfrak{A} \perp_o \mathfrak{A}_1$ and $\mathfrak{A} \perp_o \mathfrak{A}_2$

A pair consisting of an observer and a parametrization κ will be called a *parametrized observer.* In this case we will write $\mathcal{A}_\kappa(t_1, t_2) := \kappa^{-1}$ (t_1, t_2) for the inverse of κ.

Let us discuss the idea behind this definition. Axiom (i) says that associated to each pair of times t_1, t_2 there is an algebra $\mathcal{A}_\kappa(t_1, t_2)$. Self-adjoint elements of this algebra are observables the observer can observe between the times t_1 and t_2. The parametrization κ represents here the clock our observer uses. Changing the parametrization means changing the clock. It can be shown that reparametrizations are simply done by homeomorphisms of the real line (Keyl, n.d.-b).

Axiom (ii) describes together with axiom (iii) the causal structure of the observer. Let us discuss (ii) first. Since $\mathcal{T}$ is a sublattice of $\mathcal{L}^{cc}$, the smallest element of $\mathcal{T}$ which contains two algebras $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{T}$ as subalgebras equals $\sup_{\mathcal{L}^{cc}}$ $\{\mathfrak{A}_1, \mathfrak{A}_2\}$. Consider now a parametrization κ of $\mathcal{T}$ and assume that $\kappa(\mathfrak{A}_1) = (t_1,t_2)$ and $\kappa(\mathfrak{A}_2) = (t_3, t_4)$ holds with $t_1 < t_2 < t_3 < t_4$. Since κ is an order isomorphism, the supremum $\sup_{\mathcal{L}cc}\{\mathfrak{A}_1, \mathfrak{A}_2\}$ of $\mathfrak{A}_1, \mathfrak{A}_2$ in $\mathcal{T}$ is mapped by κ to (t_1, t_4). Hence we have for all t_5, t_6 with $t_2 < t_5 < t_6 < t_3$ the relation $\mathcal{A}_\kappa$ $(t_5,t_6) \subset \sup_{\mathcal{L}^{cc}}\{\mathfrak{A}_1, \mathfrak{A}_2\}$, but $\mathfrak{A}_\kappa(t_5, t_6) \perp_o \mathfrak{A}_1$ and $\mathcal{A}_\kappa(t_5, t_6) \perp_o \mathfrak{A}_2$. According to (9), we get $\neg$ $(\mathfrak{A}_1 \perp_c \mathfrak{A}_2)$. In other words, the regions $\lambda(\mathfrak{A}_1)$ and $\lambda(\mathfrak{A}_2)$ are (if λ is a causally admissible space-time realization) not spacelike-separated. However, it is still possible that subregions of $\lambda(\mathfrak{A}_1)$ and $\lambda(\mathfrak{A}_2)$ are. Physically this is not very reasonable, since each experiment we can perform between the times t_1, t_2 should be able to influence all later measurements. To see how axiom (iii) solves this problem, consider two causally closed regions $\mathbb{O}_1 \subset \lambda(\mathfrak{A}_1)$ and $\mathbb{O}_2 \subset \lambda(\mathfrak{A}_2)$. The corresponding algebras $\mathcal{A}_\lambda$ $(\mathbb{O}_1)$ and $\mathcal{A}_\lambda$ $(\mathbb{O}_2)$ are in $\mathcal{L}^{cc}_{loc}$ and due to axiom (iii) the supremum $\sup_{\mathcal{L}cc}$ $\{\mathcal{A}_\lambda(\mathbb{O}_1), \mathcal{A}_\lambda(\mathbb{O}_2)\}$ contains all $\mathcal{A}_\kappa$ (t_5, t_6) with $t_2 < t_5 < t_6 < t_3$. Applying again (9), we see that $\mathbb{O}_1$ and $\mathbb{O}_2$ are not spacelike-separated.

It is easy to check that the set of algebras $\mathscr{A}_\lambda(t_1, t_2)$, $(t_1, t_2) \subset (a, b)$ forms an observer in the just discussed sense if $\gamma\ (a, b) \subset M$ is contained in an open convex set. However, is the converse true as well? In other words: If λ is a causally admissible realization, is an observer $\mathscr{T}$ always mapped to a set $\mathscr{T}_{\lambda,\gamma} := \{\mathscr{A}_\gamma\ (t_1, t_2) \mid (t_1, t_2) \subset (a, b)\}$? To answer this question we use the following definition (Keyl, n.d.-b):

Definition 5.2. Consider an additive lattice of C*-algebras $\mathscr{L}$ and an observer $\mathscr{T} \subset \mathscr{L}$.

(a) A subset $\mathscr{P} \subset \mathscr{T}$ is called an *event* on $\mathscr{T}$ if a parametrization κ: $\mathscr{T} \to \mathscr{T}(a, b)$ and a time $t \in (a, b)$ exist such that $\mathfrak{A} \in \mathscr{P}$ holds iff $t \in \kappa(\mathfrak{A})$ is satisfied. Obviously $\mathscr{P}$ is uniquely determined by κ and t.

(b) The event $\mathscr{P}$ is *covered* by an $\mathfrak{A}_1 \in \mathscr{L}$, if there is an $\mathfrak{A} \in \mathscr{P}$ with $\mathfrak{A} \subset \mathfrak{A}_1$.

The idea behind this construction is that the set $\lambda(\mathscr{P}) := \{\lambda(\mathfrak{A}) \mid \mathfrak{A} \in \mathscr{P}\}$ should shrink to a point: $\cap_{\mathfrak{A} \in \mathscr{P}} \lambda(\mathfrak{A}) = \{p\}$. In this case $\mathscr{P}$ is covered by an algebra $\mathfrak{B} \in \mathscr{L}$ if $p \in \lambda(\mathfrak{B})$. However, this idea does not work with Definition 5.1. We need an additional, physically less motivated axiom, which we will give in the next definition (Keyl, n.d.-b).

Definition 5.3. An observer $\mathscr{T} \subset \mathscr{L}$ is called *regular* if the following additional axiom is satisfied:

(iv) If a $\mathfrak{B} \in \mathscr{L}^{cc}$ does not cover an event $\mathscr{P}$, then there is an $\mathfrak{A} \in \mathscr{P}$ with $\mathfrak{B} \perp_o \mathfrak{A}$.

For regular observers we can apply now the procedure just sketched. This leads to the following theorem (Keyl, n.d.-b).

Theorem 5.4. Consider a causally simple, additive lattice $\mathscr{L}$ of C*-algebras, a causally admissible realization λ of $\mathscr{L}$ on a space-time (M, g), and a regular observer $\mathscr{T} \subset \mathscr{L}$. There exists a continuous, timelike curve γ: $(a, b) \to M$ such that $\mathscr{T} = \{\mathscr{A}_\gamma(t_1, t_2) \mid (t_1, t_2) \subset (a, b)\}$ holds.

In other words, regular observers are (disregarding the differentiability of γ) exactly those observers which admit a worldline γ as described at the beginning of this section. However, again we have found a definition which does not make explicit reference to Lorentzian manifolds and which can therefore be applied to lattices which are not causally simple.

6. CONCLUSIONS

We have seen that it is possible to reformulate at least some concepts related to space-time in terms of lattices of algebras and it is very likely that the methods presented in this paper can be extended to yet-uncovered aspects of space-time. This concerns especially time orderings, parallel transport, and free fall.

Let us first discuss time orderings. If we have an additive lattice $\mathcal{L}$ of C*-algebras and a causally admissible realization on a time-oriented Lorentzian manifold (M, g), we can introduce a binary relation on $\mathcal{L}_{\mathrm{loc}}^{\mathrm{cc}}$ by: $\mathfrak{A}_1 \prec \mathfrak{A}_2 :\Leftrightarrow$ there is a future-pointing causal curve from a point in $\lambda(\mathfrak{A}_1)$ to a point in $\lambda(\mathfrak{A}_2)$. Note that $\prec$ is a pre-ordering, but not an ordering, since it is reflexive and transitive, but not antisymmetric. (However, *it is* antisymmetric if $\mathfrak{A}_1 \perp_o \mathfrak{A}_2$ holds, so it is in some sense nearly an ordering.) This relation is linked to $\perp_c$ defined in Theorem 4.5 by the condition $\mathfrak{A}_1 \perp \mathfrak{A}_2 \Leftrightarrow \neg(\mathfrak{A}_1 \prec \mathfrak{A}_2$ or $\mathfrak{A}_2 \prec \mathfrak{A}_1)$. Due to Theorem 4.3, $\prec$ depends only on the time ordering of $(\mathcal{M}, g)$, not on the realization λ. Hence it is interesting to ask how such a relation can be introduced without the help of causally admissible realizations, and which conditions $\mathcal{L}$ has to satisfy, such that it exists.

To treat parallel transport, observers are, due to their close relation to timelike curves, a good prerequisite. Hence, consider a regular observer $\mathcal{T} \subset \mathcal{L}$ with worldline γ and the local algebra $\mathcal{A}_\gamma\,(t_1 - \varepsilon, t_1 + \varepsilon)$. Due to the interpretation given in Section 5, a self-adjoint $A_1 \in \mathcal{A}_\gamma\,(t_1 - \varepsilon, t_1 + \varepsilon)$ describes an observable measurable by the observer in the time interval $(t_1 - \varepsilon, t_1 + \varepsilon)$. Obviously the corresponding measurement is done by a certain measuring procedure which can be repeated at a later time. This leads to a second observable $A_2 \in \mathcal{A}_\gamma\,(t_2 - \varepsilon, t_2 + \varepsilon)$. Hence we have for all t_1, t_2 (at least if $t_2 > t_1$) a map $\alpha_{t_1 t_2}^{\gamma}$: $\mathcal{A}_\gamma(t_1 - \varepsilon, t_1 + \varepsilon) \to \mathcal{A}_\gamma(t_2 - \varepsilon, t_2, + \varepsilon)$ such that $\alpha_{t_1,t_2}^{\gamma}\,(A_1) = A_2$. It is very likely that these maps are related to the field equations on which the theory is based and that they depend therefore on the parallel transport along γ. In addition we can interpret the $\alpha_{t_1,t_2}^{\gamma}$ as some kind of parallel transport of local observables $A_1 \in \mathcal{A}_\gamma\,(t_1 - \varepsilon, t_1 + \varepsilon)$ along γ. If this interpretation is correct, it should be possible to characterize geodesics (i.e., worldlines of observers in free fall) by special properties of the $\alpha_{t_1,t_2}^{\gamma}$.

Apart from these open questions, we have structures which are capable of describing more general space-time concepts than Lorentzian manifolds, since the definitions derived in Sections 4 and 5 make sense even if the lattice considered cannot be related to a Lorentzian manifold by causally admissible realization. Of course, it is very likely that the discussion of this paper is still too special and too strongly related to classical space-time. However, to get a reasonable description of space-time and quantum field theory in a unique axiomatic scheme it is possible to further generalize our constructions. For example, many considerations about quantum gravity indicate that locality of quantum fields gets lost completely. This implies for us that the relation $\perp_a$ used to define $\perp_c$ is no longer useful. In this case it natural to investigate how locality is violated and how this violation can be interpreted physically. On the mathematical level this means to consider the set $\{[\mathfrak{A}_1, \mathfrak{A}_2] \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{L}\}$ of commutator algebras associated to a lattice $\mathcal{L}$ of algebras. A detailed study of particular models for quantum gravity (or at least toy models, since realistic ones are at the time unavailable) may lead on this basis to a generalization of the structures discussed in this paper.

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